

INVARIANT FOLIATIONS NEAR NORMALLY HYPERBOLIC EQUILIBRIA FOR QUASILINEAR PARABOLIC PROBLEMS

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ABSTRACT. We consider quasilinear parabolic evolution equations in the situation where the set of equilibria forms a finite-dimensional C^1 -manifold which is normally hyperbolic. The existence of foliations of the stable and unstable manifolds is shown assuming merely C^1 -regularity of the underlying equation.

Keywords: Quasilinear parabolic equations, normally stable, normally hyperbolic, invariant foliations, free boundary problems, symmetry.

AMS subject classification: 35K55, 35B35, 37D10, 37D30 35R35, 34G20.

1. INTRODUCTION

In recent years, foliations of invariant manifolds of (semi)-flows have attracted considerable attention. They exhibit a precise analysis of the given (semi)-flow near a certain special invariant manifold \mathcal{E} , most often a center manifold. We refer to [1, 2, 3] for a comprehensive discussion of the subject for infinite-dimensional dynamical systems, and to [10, 21] for the finite-dimensional case.

In this paper we consider a special situation, namely the case where \mathcal{E} is a C^1 -manifold of equilibria for a quasilinear parabolic system. The precise setting for this paper is as follows. Let X_0 and X_1 be two Banach spaces such that X_1 is continuously and densely embedded in X_0 . We then consider the abstract autonomous quasilinear problem

$$\dot{u}(t) + A(u(t))u(t) = F(u(t)), \quad t > 0, \quad u(0) = u_0. \quad (1.1)$$

For $1 < p < \infty$ we introduce the real interpolation space $X_\gamma := (X_0, X_1)_{1-1/p, p}$ and we assume that there is an open set $V \subset X_\gamma$ such that

$$(A, F) \in C^1(V, \mathcal{B}(X_1, X_0) \times X_0). \quad (1.2)$$

Here $\mathcal{B}(X_1, X_0)$ denotes the space of all bounded linear operators from X_1 into X_0 . In the sequel we use the notation $|\cdot|_j$ to denote the norm in the respective spaces X_j for $j \in \{0, 1, \gamma\}$. Moreover, for any normed space X , $B_X(u, r)$ denotes the open ball in X with radius $r > 0$ around $u \in X$.

If $A(u)$ has the property of **maximal L_p -regularity** for each $u \in V$, then it is well-known that problem (1.1) generates a local semiflow in $V \subset X_\gamma$; see e.g.

[11, 13]. Let $\mathcal{E} \subset V \cap X_1$ denote the set of equilibrium solutions of (1.1), which means that

$$u \in \mathcal{E} \quad \text{if and only if} \quad u \in V \cap X_1, \quad A(u)u = F(u).$$

Given an element $u_* \in \mathcal{E}$, we consider the situation that u_* is contained in an m -dimensional manifold of equilibria. This means that there is an open subset $U \subset \mathbb{R}^m$, $0 \in U$, and a C^1 -function $\Psi : U \rightarrow X_1$, such that

- $\Psi(U) \subset \mathcal{E}$ and $\Psi(0) = u_*$,
 - the rank of $\Psi'(0)$ equals m , and
 - $A(\Psi(\zeta))\Psi(\zeta) = F(\Psi(\zeta)), \quad \zeta \in U.$
- (1.3)

Let $A_0 := A(u_*) + [A'(u_*) \cdot]u_* - F'(u_*)$ be the linearization of (1.1) at $u_* \in \mathcal{E}$. An equilibrium u_* is called **normally hyperbolic** if

- (i) near u_* the set of equilibria \mathcal{E} is a C^1 -manifold in X_1 of dimension $m \in \mathbb{N}$,
- (ii) the tangent space for \mathcal{E} at u_* is given by $N(A_0)$,
- (iii) 0 is a semi-simple eigenvalue of A_0 , i.e. $N(A_0) \oplus R(A_0) = X_0$,
- (iv) $\sigma(A_0) \cap i\mathbb{R} = \{0\}$, $\sigma_u := \sigma(A_0) \cap \mathbb{C}_- \neq \emptyset$.

An equilibrium is called **normally stable** if (i), (ii), (iii) hold and (iv) is replaced by

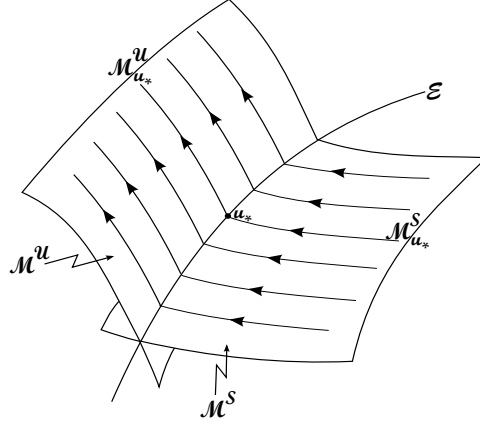
$$(iv)_s \quad \sigma(A_0) \cap i\mathbb{R} = \{0\}, \quad \sigma(A_0) \setminus \{0\} \subset \mathbb{C}_+,$$

which means that $\sigma_u = \emptyset$. Note that a standard argument in the theory of Fredholm operators implies that $R(A_0)$ is closed and A_0 is a Fredholm operator of index 0, even if (iii) holds only algebraically, see for instance [4, p.78]. So in fact (iii) is also true topologically, and $\{0\}$ is an isolated point in $\sigma(A_0)$.

In the recent paper [16] it is proved that any solution $u(t)$ starting near a normally hyperbolic $u_* \in \mathcal{E}$ and staying near \mathcal{E} exists globally and converges in X_γ to an equilibrium $u_\infty \in \mathcal{E}$ which, however, may be different from u_* . In case u_* is normally stable this is true for any solution starting near u_* . We call this the **generalized principle of linearized stability**. In that paper it is also shown that near u_* there are no other equilibria than those given by $\Psi(U)$, i.e. $\mathcal{E} \cap B_{X_1}(u_*, r_1) = \Psi(U)$, for some $r_1 > 0$.

These situations appear frequently in applications, for example in problems with symmetries, and in problems with moving boundaries, see for instance [8, 9, 12, 14, 15, 16, 17, 18, 19, 20].

Our intention in this paper is to study the behavior of the semiflow near \mathcal{E} in more detail. If u_* is normally hyperbolic, then any $w \in \mathcal{E}$ close to u_* in X_γ will be normally hyperbolic as well. Therefore, intuitively, at each point $w \in \mathcal{E}$ near u_* there should be a stable manifold \mathcal{M}_w^s and an unstable manifold \mathcal{M}_w^u such that $\mathcal{M}_w^s \cap \mathcal{M}_w^u \cap B(u_*, r) = \{w\}$, and these manifolds should depend continuously on $w \in B(u_*, r) \cap \mathcal{E}$. The tangent spaces of these manifolds are expected to be the projections with respect to the stable resp. unstable part of the spectrum of the linearization of (1.1) at w .

FIGURE 1. Foliation and fibers near \mathcal{E} .

We prove these assertions below, and call it the **stable** resp. **unstable foliation** of (1.1) near $u_* \in \mathcal{E}$. The stable resp. unstable manifolds \mathcal{M}_w^s and \mathcal{M}_w^u are termed the **leaves** or **fibers** of these foliations. They turn out to be positively and also negatively invariant under the semiflow. As a consequence, the convergent solutions are precisely those which start at initial values sitting on one of the stable fibers. In the normally stable case where $\sigma(A_0) \cap \mathbb{C}_- = \emptyset$, this implies that the stable foliation covers a neighborhood of u_* in V .

The proofs we present here are quite simple and elementary, as they just employ the implicit function theorem, which is at our disposal thanks to maximal regularity. We only need to require C^1 -regularity for (A, F) and for \mathcal{E} , in contrast to other work, where more regularity (at least C^2) is assumed. We mention that the authors in [2] announced a proof for the C^1 case in a general context, see page 4644.

The fibers of these foliations will be manifolds of class C^1 as well, however, their dependence on $w \in \mathcal{E}$ is only continuous: here we lose one degree of regularity. This is quite natural and should be compared to the loss of regularity which the normal field ν of a hyper-surface $\Gamma \subset \mathbb{R}^n$ of class C^k , $k \in \mathbb{N}$, suffers: it is only of class C^{k-1} . We also show that in case $(A, F) \in C^k$ the dependence on w is of class C^{k-1} , for each $k \in \mathbb{N} \cup \{\infty, \omega\}$, where ω means real analytic.

2. PRELIMINARIES AND THE ASYMPTOTIC NORMAL FORM

(a) Suppose that the operator $A(u_*)$ has the property of maximal L_p -regularity. Introducing the deviation $v = u - u_*$ from the equilibrium u_* , the equation for v then reads as

$$\dot{v}(t) + A_0 v(t) = G(v(t)), \quad t > 0, \quad v(0) = v_0, \quad (2.1)$$

where $v_0 = u_0 - u_*$ and

$$A_0 w = A(u_*)w + (A'(u_*)w)u_* - F'(u_*)w \quad \text{for } w \in X_1. \quad (2.2)$$

The function G can be written as $G(v) = G_1(v) + G_2(v, v)$, where

$$\begin{aligned} G_1(v) &= (F(u_* + v) - F(u_*) - F'(u_*)v) - (A(u_* + v) - A(u_*) - A'(u_*)v)u_*, \\ G_2(v, w) &= -(A(u_* + v) - A(u_*))w, \quad w \in X_1, v \in V_*, \end{aligned}$$

and $V_* := V - u_*$. It follows from (1.2) that $G_1 \in C^1(V_*, X_0)$ and also that $G_2 \in C^1(V_* \times X_1, X_0)$. Moreover, we have

$$G_1(0) = G_2(0, 0) = 0, \quad G'_1(0) = G'_2(0, 0) = 0, \quad (2.3)$$

where G'_1 and G'_2 denote the Fréchet derivatives of G_1 and G_2 , respectively.

Setting $\psi(\zeta) = \Psi(\zeta) - u_*$ results in the following equilibrium equation for problem (2.1):

$$A_0 \psi(\zeta) = G(\psi(\zeta)), \quad \text{for all } \zeta \in U. \quad (2.4)$$

Taking the derivative with respect to ζ and using the fact that $G'(0) = 0$ we conclude that $A_0 \psi'(0) = 0$ and this implies that

$$T_{u_*}(\mathcal{E}) \subset N(A_0), \quad (2.5)$$

where $T_{u_*}(\mathcal{E})$ denotes the tangent space of \mathcal{E} at u_* . Note that assumption (ii) yields equality in (2.5), and (iii) implies that 0 is an isolated spectral point of A_0 . According to assumption (iv), $\sigma(A_0)$ admits a decomposition into three disjoint parts with

$$\sigma(A_0) = \{0\} \cup \sigma_s \cup \sigma_u, \quad -\sigma_u \cup \sigma_s \subset \mathbb{C}_+ = \{z \in \mathbb{C} : \operatorname{Re} z > 0\}.$$

The spectral set $\sigma_c := \{0\}$ corresponds to the center part, σ_s to the stable one, and σ_u to the unstable part of the analytic C_0 -semigroup $\{e^{-A_0 t} : t \geq 0\}$, or equivalently of the Cauchy problem $\dot{w} + A_0 w = f$.

In the normally stable case we have $\sigma_u = \emptyset$. Subsequently, we let P^l , $l \in \{c, s, u\}$, denote the spectral projections according to the spectral sets $\sigma_c = \{0\}$, σ_s , and σ_u , and we set $X_j^l := P^l X_j$ for $l \in \{c, s, u\}$ and $j \in \{0, 1, \gamma\}$. The spaces X_j^l are equipped with the norms $|\cdot|_j$ for $j \in \{0, 1, \gamma\}$. We have the topological direct decomposition

$$X_1 = X_1^c \oplus X_1^s \oplus X_1^u, \quad X_0 = X_0^c \oplus X_0^s \oplus X_0^u,$$

and this decomposition reduces A_0 into $A_0 = A_c \oplus A_s \oplus A_u$, where A_l is the part of A_0 in X_0^l for $l \in \{c, s, u\}$. Since $\sigma_c = \{0\}$ and σ_u is compact it follows that $X_0^c \subset X_1$ and $X_0^u \subset X_1$. Therefore, X_0^l and X_1^l coincide as vector spaces for $l \in \{c, u\}$. In the following, we will just write $X^l = (X^l, |\cdot|_l)$ for either of the spaces X_0^l and X_1^l , $l \in \{c, u\}$. We note that $X^c = N(A_0)$, and hence $A_c = 0$.

The operator A_s inherits the property of L_p -maximal regularity from A_0 . Since $\sigma(A_s) = \sigma_s \subset \mathbb{C}_+$, we obtain that the Cauchy problem

$$\dot{w} + A_s w = f, \quad w(0) = 0, \quad (2.6)$$

also enjoys the property of maximal regularity, even on the interval $J = (0, \infty)$. In fact the following estimates are true: for any $a \in (0, \infty]$ let

$$\mathbb{E}_0(a) = L_p((0, a); X_0), \quad \mathbb{E}_1(a) = H_p^1((0, a); X_0) \cap L_p((0, a); X_1). \quad (2.7)$$

The natural norms in $\mathbb{E}_j(a)$ will be denoted by $\|\cdot\|_{\mathbb{E}_j(a)}$ for $j = 0, 1$. Then the Cauchy problem (2.6) has for each $f \in L_p((0, a); X_0^s)$ a unique solution

$$w \in H_p^1((0, a); X_0^s) \cap L_p((0, a); X_1^s),$$

and there exists a constant M_0 such that $\|w\|_{\mathbb{E}_1(a)} \leq M_0 \|f\|_{\mathbb{E}_0(a)}$ for every $a > 0$, and every function $f \in L_p((0, a); X_0^s)$. In fact, since $\sigma(A_s - \omega)$ is still contained in \mathbb{C}_+ for

$$0 < \omega < \inf \operatorname{Re} \sigma_s$$

fixed, we see that the operator $(A_s - \omega)$ enjoys the same properties as A_s . Therefore, every solution of the Cauchy problem (2.6) satisfies the estimate

$$\|e^{\sigma t} w\|_{\mathbb{E}_1(a)} \leq M_0 \|e^{\sigma t} f\|_{\mathbb{E}_0(a)}, \quad \sigma \in [0, \omega], \quad a > 0, \quad (2.8)$$

for $f \in L_p((0, a); X_0^s)$, where $M_0 = M_0(\omega)$ for $\omega > 0$ chosen as above, see [7, Theorem 2.4] or [13, Sec. 6]. Furthermore, there exists a constant $M_1 > 0$ such that

$$\|e^{\omega t} e^{-A_s t} P^s u\|_{\mathbb{E}_1(a)} + \sup_{t \in [0, a]} |e^{\omega t} e^{-A_s t} P^s u|_\gamma \leq M_1 |P^s u|_\gamma \quad (2.9)$$

for every $u \in X_\gamma$ and $a \in (0, \infty]$. For future use we note that

$$\sup_{t \in [0, a]} |w(t)|_\gamma \leq c_0 \|w\|_{\mathbb{E}_1(a)} \quad \text{for all } w \in \mathbb{E}_1(a) \text{ with } w(0) = 0 \quad (2.10)$$

with a constant c_0 that is independent of $a \in (0, \infty]$.

In the following we assume that

$$0 < \omega < \min\{\inf \operatorname{Re} \sigma_s, -\sup \operatorname{Re} \sigma_u\}.$$

Turning our attention to A_u , we observe that A_u is bounded, and hence e^{-tA_u} extends to a group $\{e^{-A_u t} : t \in \mathbb{R}\}$ on X^u which satisfies the estimate

$$|e^{-A_u t}| \leq M_\omega e^{\omega t}, \quad t \leq 0. \quad (2.11)$$

We emphasize that L_p -maximal regularity of $A(u_*)$ implies by standard perturbation theory that $A(u)$ has this property as well, for each u in some ball $\bar{B}_{X_\gamma}(u_*, \rho_0)$, and so all estimates in this paragraph are uniform w.r.t. $u \in \bar{B}_{X_\gamma}(u_*, \rho_0)$.

(b) Let us now consider the mapping

$$g : U \subset \mathbb{R}^m \rightarrow X^c, \quad g(\zeta) := P^c \psi(\zeta), \quad \zeta \in U.$$

It follows from our assumptions that $g'(0) = P^c \psi'(0) : \mathbb{R}^m \rightarrow X^c$ is an isomorphism (between the finite dimensional spaces \mathbb{R}^m and X^c). By the inverse function theorem, g is a C^1 -diffeomorphism of a neighborhood of 0 in \mathbb{R}^m into a neighborhood, say $B_{X^c}(0, \rho_0)$, of 0 in X^c . Let $g^{-1} : B_{X^c}(0, \rho_0) \rightarrow U$ be the inverse mapping.

Then $g^{-1} : B_{X^c}(0, \rho_0) \rightarrow U$ is C^1 and $g^{-1}(0) = 0$. Next we set $\Phi(x) := \psi(g^{-1}(x))$ for $x \in B_{X^c}(0, \rho_0)$ and we note that

$$\Phi \in C^1(B_{X^c}(0, \rho_0), X_1), \quad \Phi(0) = 0, \quad \{\Phi(x) + u_* : x \in B_{X^c}(0, \rho_0)\} = \mathcal{E} \cap W,$$

where W is an appropriate neighborhood of u_* in X_1 . One readily verifies that

$$P^c \Phi(x) = ((P^c \circ \psi) \circ g^{-1})(x) = (g \circ g^{-1})(x) = x, \quad x \in B_{X^c}(0, \rho_0),$$

and this yields $\Phi(x) = P^c \Phi(x) + P^s \Phi(x) + P^u \Phi(x) = x + P^s \Phi(x) + P^u \Phi(x)$ for $x \in B_{X^c}(0, \rho_0)$. Setting $\phi_l(x) := P^l \Phi(x)$, $l \in \{s, u\}$, we conclude that

$$\phi_j \in C^1(B_{X^c}(0, \rho_0), X_1^j), \quad \phi_j(0) = \phi'_j(0) = 0, \quad (2.12)$$

and with $\phi := \phi_s + \phi_u$ that

$$\{x + \phi(x) + u_* : x \in B_{X^c}(0, \rho_0)\} = \mathcal{E} \cap W.$$

This shows that the manifold \mathcal{E} can be represented as the (translated) graph of the function ϕ in a neighborhood of u_* . Moreover, the tangent space of \mathcal{E} at u_* coincides with $N(A_0) = X^c$. By applying the projections P^l , $l \in \{c, s, u\}$, to equation (2.4) and using that $x + \phi(x) = \psi(g^{-1}(x))$ for $x \in B_{X^c}(0, \rho_0)$, and that $A_c \equiv 0$, we obtain the following equivalent system of equations for the equilibria of (2.1):

$$\begin{aligned} P^c G(x + \phi(x)) &= 0, \\ P^l G(x + \phi(x)) &= A_l \phi_l(x), \quad x \in B_{X^c}(0, \rho_0), \quad l \in \{s, u\}. \end{aligned} \quad (2.13)$$

(c) Introducing the new variables

$$\begin{aligned} x &= P^c v - \xi = P^c(u - u_*) - \xi, \\ y &= P^s v - \phi_s(\xi) = P^s(u - u_*) - \phi_s(\xi), \\ z &= P^u v - \phi_u(\xi) = P^u(u - u_*) - \phi_u(\xi), \end{aligned}$$

where $\xi \in B_{X^c}(0, \rho_0)$, we obtain the following system of evolution equations in $X^c \times X_0^s \times X^u$:

$$\begin{cases} \dot{x} = R_c(x, y, z, \xi), & x(0) = x_0 - \xi, \\ \dot{y} + A_s y = R_s(x, y, z, \xi), & y(0) = y_0 - \phi_s(\xi), \\ \dot{z} + A_u z = R_u(x, y, z, \xi), & z(0) = z_0 - \phi_u(\xi), \end{cases} \quad (2.14)$$

with $x_0 = P^c v_0$, $y_0 = P^s v_0$, $z_0 = P^u v_0$, and

$$\begin{aligned} R_c(x, y, z, \xi) &= P^c G(x + y + z + \xi + \phi(\xi)), \\ R_l(x, y, z, \xi) &= P^l G(x + y + z + \xi + \phi(\xi)) - A_l \phi_l(\xi) \end{aligned}$$

for $l \in \{s, u\}$. Using the equilibrium equations (2.13), the expressions for R_l can be rewritten as

$$R_l(x, y, z, \xi) = P^l(G(x + y + z + \xi + \phi(\xi)) - G(\xi + \phi(\xi))), \quad (2.15)$$

where $l \in \{c, s, u\}$. Although the term $P^c G(\xi + \phi(\xi))$ in R_c is zero, see (2.13), we include it here for reasons of symmetry. Equation (2.15) immediately yields

$$R_c(0, 0, 0, \xi) = R_s(0, 0, 0, \xi) = R_u(0, 0, 0, \xi) = 0 \quad \text{for all } \xi \in B_{X^c}(0, \rho_0),$$

showing that the equilibrium set \mathcal{E} of (1.1) near u_* has been reduced to the set $\{0\} \times \{0\} \times \{0\} \times B_{X^c}(0, \rho_0) \subset X^c \times X_1^s \times X^u \times X^c$.

Observe also that there is a unique correspondence between the solutions of (1.1) close to u_* in X_γ and those of (2.14) close to 0. As $u_\infty := u_* + \xi + \phi(\xi) \in \mathcal{E}$ will be the limit of $u(t)$ in X_γ as $t \rightarrow \infty$, we call system (2.14) the *asymptotic normal form* of (1.1) near its normally stable equilibrium u_* .

3. THE STABLE FOLIATION

To motivate our approach to the construction of the stable foliation we formally define a map H_s according to

$$H_s((x, y, z), (y_0, \xi))(t) = \begin{bmatrix} x(t) + \int_t^\infty R_c(x(\tau), y(\tau), z(\tau), \xi) d\tau \\ y(t) - L_s(R_s(x, y, z, \xi), y_0 - \phi_s(\xi)) \\ z(t) + \int_t^\infty e^{-A_u(t-\tau)} R_u(x(\tau), y(\tau), z(\tau), \xi) d\tau \end{bmatrix}, \quad (3.1)$$

where $t > 0$. Here $w = L_s(f, y_0)$ denotes the unique solution of the problem

$$\dot{w}(t) + A_s w(t) = f(t), \quad t > 0, \quad w(0) = y_0.$$

Obviously, we have $H_s(0, 0) = 0$. Moreover, H_s will be of class C^1 w.r.t. to the variables (x, y, z, y_0) , but in general only continuous in ξ . The derivative of H_s w.r.t. (x, y, z) at $(0, 0)$ is given by $D_{(x, y, z)} H_s(0, 0) = I$, and hence the implicit function theorem formally applies and yields a map

$$\Lambda^s : (y_0, \xi) \mapsto (x, y, z) \quad (3.2)$$

which is well-defined near 0, such that $H_s(\Lambda^s(y_0, \xi), (y_0, \xi)) = 0$. Λ^s will be continuous in (y_0, ξ) , and of class C^1 w.r.t. y_0 . Given $(x, y, z) = \Lambda^s(y_0, \xi)$, we set

$$\begin{aligned} x_0 &:= - \int_0^\infty R_c(x(\tau), y(\tau), z(\tau), \xi) d\tau + \xi, \\ z_0 &:= - \int_0^\infty e^{A_u \tau} R_u(x(\tau), y(\tau), z(\tau), \xi) d\tau + \phi_u(\xi). \end{aligned} \quad (3.3)$$

For $(x, y, z) = \Lambda^s(y_0, \xi)$ given, the first component of H_s yields

$$\dot{x}(t) = R_c(x(t), y(t), z(t), \xi), \quad t \geq 0,$$

the second component implies

$$\dot{y}(t) + A_s y(t) = R_s(x(t), y(t), z(t), \xi), \quad t > 0, \quad y(0) = y_0 - \phi_s(\xi),$$

and the third one leads to

$$\dot{z}(t) + A_u z(t) = R_u(x(t), y(t), z(t), \xi), \quad t > 0.$$

Moreover, due to (3.3), the initial values of x and z are given by

$$x(0) = x_0 - \xi, \quad z(0) = z_0 - \phi_u(\xi).$$

Assuming, in addition, that $(x(t), y(t), z(t))$ converges to $(0, 0, 0)$ as $t \rightarrow \infty$, we conclude that

$$u(t) = u_* + x(t) + y(t) + z(t) + \xi + \phi(\xi), \quad t > 0,$$

is a solution of (1.1) with $\lim_{t \rightarrow \infty} u(t) = u_\infty := u_* + \xi + \phi(\xi) \in \mathcal{E}$. The map

$$\lambda^s : (y_0, \xi) \mapsto u(0) = u_* + x(0) + y(0) + z(0) + \xi + \phi(\xi) \quad (3.4)$$

yields a foliation of the stable manifold \mathcal{M}^s near u_* , and

$$\mathcal{M}_\xi^s := \{\lambda^s(y_0, \xi) : y_0 \in B_{X^s}(0, r)\} \quad (3.5)$$

are the fibers over $B_{X^c}(0, r)$, or equivalently over \mathcal{E} near u_* . We note that the fibers are C^1 -manifolds, but they will depend only continuously on ξ , or equivalently on \mathcal{E} .

The strategy of our approach can be summarized as follows: given a base point $u_* + \xi + \phi(\xi)$ on the manifold \mathcal{E} , and an initial value $y_0 \in X_\gamma^s$, we determine with the help of the implicit function theorem an initial value u_0 and a solution $u(t)$ such that $u(t)$ converges to the base point $u_* + \xi + \phi(\xi)$ exponentially fast. Exponential convergence will be obtained by setting up the implicit function theorem in a space of exponentially decaying functions.

After these heuristic considerations we can state our first main result, employing the notation introduced above.

Theorem 3.1. *Consider (1.1) under the assumption (1.2), and let $u_* \in \mathcal{E}$ be a normally hyperbolic equilibrium. Suppose further that $A(u_*)$ has the property of L_p -maximal regularity. Then there is a number $r > 0$ and a continuous map*

$$\lambda^s : B_{X_\gamma^s}(0, r) \times B_{X^c}(0, r) \rightarrow X_\gamma \quad \text{with} \quad \lambda^s(0, 0) = u_*,$$

*the **stable foliation**, such that the solution $u(t)$ of (1.1) with initial value $\lambda^s(y_0, \xi)$ exists on \mathbb{R}_+ and converges to $u_\infty := u_* + \xi + \phi(\xi)$ in X_γ exponentially fast as $t \rightarrow \infty$. The image of λ^s defines the stable manifold \mathcal{M}^s of (1.1) near u_* .*

Furthermore, for fixed $\xi \in B_{X^c}(0, r)$, the function

$$\lambda_\xi^s : B_{X_\gamma^s}(0, r) \rightarrow X_\gamma, \quad \text{given by} \quad \lambda_\xi^s(y_0) = \lambda^s(y_0, \xi),$$

defines the fibers $\mathcal{M}_\xi^s := \lambda_\xi^s(B_{X_\gamma^s}(0, r))$ of the foliation. Moreover, we have

- (i) *an initial value $u_0 \in X_\gamma$ near u_* belongs to \mathcal{M}^s if and only if the solution $u(t)$ of (1.1) exists globally on \mathbb{R}_+ and converges to some $u_\infty \in \mathcal{E}$ exponentially fast as $t \rightarrow \infty$;*
- (ii) *λ_ξ^s is of class C^1 , and the derivative $D_{y_0} \lambda^s$ is continuous, jointly in (y_0, ξ) ;*
- (iii) *the fibers are C^1 -manifolds which are invariant under the semiflow generated by (1.1);*

- (iv) the tangent space of $\mathcal{M}_{u_\infty}^s$ at $u_\infty \in \mathcal{E}$ is precisely the projection of the stable part of the linearization of (1.1) at u_∞ ;
- (v) if u_* is normally stable then \mathcal{M}^s forms a neighborhood of u_* in X_γ .

Proof. For fixed $\sigma \in (0, \omega]$ we define the function spaces

$$\begin{aligned}\mathbb{F}_0^l(\sigma) &:= \{f : e^{\sigma t} f \in L_p(\mathbb{R}_+; X_0^l)\}, \\ \mathbb{F}_1^l(\sigma) &:= \{w : e^{\sigma t} w \in H_p^1(\mathbb{R}_+; X_0^l) \cap L_p(\mathbb{R}_+; X_1^l)\}, \quad l \in \{c, s, u\},\end{aligned}$$

equipped with their natural norms. Then we set

$$\mathbb{Y} := \mathbb{Y}(\sigma) := \mathbb{F}_1^c(\sigma) \times \mathbb{F}_1^s(\sigma) \times \mathbb{F}_1^u(\sigma), \quad Z := X_\gamma^s \times X^c,$$

and we define $H_s : B_{\mathbb{Y}}(0, \rho) \times B_Z(0, \rho) \rightarrow \mathbb{Y}$ by (3.1). Observe that H_s is the composition of the substitution operator

$$R(x, y, z, \xi) = [(R_c, R_s, R_u)(x + y + z + \xi + \phi(\xi)), \phi_s(\xi)]^\top,$$

which maps $B_{\mathbb{Y}}(0, \rho) \times B_Z(0, \rho)$ into $\mathbb{F}_0^c(\sigma) \times \mathbb{F}_0^s(\sigma) \times \mathbb{F}_0^u(\sigma) \times X_\gamma^s$, and the bounded linear operator

$$\mathbb{L}(x, y, z, y_0, \xi, R_1, R_2, R_3, R_4) = \begin{bmatrix} x(t) + \int_t^\infty R_1(\tau) d\tau \\ y(t) - L_s(R_2, y_0 - R_4) \\ z(t) + \int_t^\infty e^{-A_u(t-\tau)} R_3(\tau) d\tau \end{bmatrix},$$

which maps

$$\mathbb{Y} \times Z \times \mathbb{F}_0^c(\sigma) \times \mathbb{F}_0^s(\sigma) \times \mathbb{F}_0^u(\sigma) \times X_\gamma^s \quad \text{into} \quad \mathbb{Y}.$$

In order to see that the integral operator $[R_1 \mapsto \int_t^\infty R_1(\tau) d\tau]$ maps $\mathbb{F}_0^c(\sigma)$ into $\mathbb{F}_1^c(\sigma)$ we set

$$(KR_1)(t) := \int_t^\infty R_1(\tau) d\tau, \quad R_1 \in \mathbb{F}_0^c(\sigma).$$

Clearly, $e^{\delta t}(Kg)(t) = \int_t^\infty e^{\delta(t-\tau)} e^{\delta\tau} g(\tau) d\tau$, and Young's inequality for convolution integrals readily yields

$$K \in \mathcal{B}(\mathbb{F}_0^c(\sigma), \mathbb{F}_1^c(\sigma)).$$

Similar arguments also apply for the term $\int_t^\infty e^{-A_u(t-\tau)} R_3(\tau) d\tau$.

As G is of class C^1 , it is not difficult to see that R is C^1 with respect to the variables (x, y, z, y_0) , but in general only continuous with respect to ξ . This, in turn, implies that H_s is of class C^1 w.r.t. (x, y, z, y_0) , and continuous in ξ . In addition, as $G(0) = G'(0) = 0$, we obtain

$$H_s(0, 0) = 0, \quad D_{(x, y, z)} H_s(0, 0) = I_{\mathbb{Y}}.$$

Therefore, by the implicit function theorem, see [4, Theorem 15.1], there is a radius $r > 0$ and a continuous map $\Lambda^s : B_Z(0, r) \rightarrow \mathbb{Y}$ such that

$$H_s(\Lambda^s(y_0, \xi), (y_0, \xi)) = 0, \quad \text{for all } (y_0, \xi) \in B_Z(0, r),$$

and there is no other solution of $H_s((x, y, z), (y_0, \xi)) = 0$ in the ball $B_Y(0, r) \times B_Z(0, r)$. Moreover, Λ^s is also C^1 w.r.t. y_0 , and one shows that

$$D_{y_0}\Lambda^s(0, 0)w_0 = [0, e^{-tA_s}w_0, 0]^\top. \quad (3.6)$$

We may now continue as indicated in the heuristic considerations preceding Theorem 3.1 to define the stable manifold \mathcal{M}^s and their fibers \mathcal{M}_ξ^s near u_* , see (3.4)–(3.5). Clearly, the fibers are positively and negatively invariant under the semiflow, as (1.1) is invariant concerning time-translations. As λ_ξ^s is of class C^1 , it is clear that the fibers are C^1 -manifolds, parameterized over X_γ . In fact, the fibers are diffeomorphic, which can be seen by interchanging the roles of u_* and $u_\infty = u_* + \xi + \phi(\xi)$.

It follows from (3.6) that $D_{y_0}\lambda^s(0, 0) = I_{X_\gamma^s}$. Interchanging the roles of u_* and u_∞ , it becomes clear that the tangent space of the fiber \mathcal{M}_ξ^s at 0 is precisely the projection of the stable part of the linearization $A_\infty = A(u_\infty) + [A'(u_\infty) \cdot]u_\infty - F'(u_\infty)$ of (1.1) at $u_\infty \in \mathcal{E}$, yielding assertion (iv).

To obtain the characterization of \mathcal{M}^s , observe that we proved in [16] that there are balls $B_{X_\gamma}(u_*, r_0)$ and $B_{X_\gamma}(u_*, r_1)$ such that any solution of (1.1) starting in $B_{X_\gamma}(u_*, r_0)$ and staying near \mathcal{E} stays in $B_{X_\gamma}(u_*, r_1)$ and converges to an equilibrium $u_\infty \in \mathcal{E}$ exponentially fast. This implies that its initial value must belong to \mathcal{M}^s by uniqueness of Λ^s .

If $u_* \in \mathcal{E}$ is normally stable then, as proved in [16], there are balls $B_{X_\gamma}(u_*, r_0)$ and $B_{X_\gamma}(u_*, r_1)$ such that any solution of (1.1) starting in $B_{X_\gamma}(u_*, r_0)$ stays in $B_{X_\gamma}(u_*, r_1)$ and converges to an equilibrium $u_\infty \in \mathcal{E}$ exponentially fast. This implies that \mathcal{M}^s forms a neighborhood of u_* in X_γ , thus establishing assertion (v) of the theorem. \square

4. THE UNSTABLE FOLIATION

Our second main result concerning the unstable foliation of \mathcal{E} reads as follows.

Theorem 4.1. *Consider (1.1) under the assumption (1.2), and let $u_* \in \mathcal{E}$ be a normally hyperbolic equilibrium. Suppose further that $A(u_*)$ has the property of L_p -maximal regularity. Then there is a number $r > 0$ and a continuous map*

$$\lambda^u : B_{X_\gamma^u}(0, r) \times B_{X^c}(0, r) \rightarrow X_\gamma \quad \text{with} \quad \lambda^u(0, 0) = u_*,$$

*the **unstable foliation**, such that the solution $u(t)$ of (1.1) with initial value $\lambda^u(y_0, \xi)$ exists on \mathbb{R}_- and converges to $u_\infty := u_* + \xi + \phi(\xi)$ in X_γ exponentially fast as $t \rightarrow -\infty$. The image of λ^u defines the unstable manifold \mathcal{M}^u of (1.1) near u_* .*

Furthermore, for fixed $\xi \in B_{X^c}(0, r)$, the function

$$\lambda_\xi^u : B_{X_\gamma^u}(0, r) \rightarrow X_\gamma, \quad \text{given by} \quad \lambda_\xi^u(y_0) = \lambda^u(y_0, \xi),$$

defines the fibers $\mathcal{M}_\xi^u := \lambda_\xi^u(B_{X_\gamma^u}(0, r))$ of the foliation. Moreover, we have

- (i) an initial value $u_0 \in X_\gamma$ near u_* belongs to \mathcal{M}^u if and only if the solution $u(t)$ of (1.1) exists globally on \mathbb{R}_- and converges to some $u_\infty \in \mathcal{E}$ exponentially fast as $t \rightarrow -\infty$;
- (ii) λ_ξ^u is of class C^1 , and the derivative $D_{y_0}\lambda^u$ is continuous, jointly in (y_0, ξ) ;
- (iii) the fibers are C^1 -manifolds which are invariant under the semiflow generated by (1.1);
- (iv) the tangent space of \mathcal{M}_∞^u at $u_\infty \in \mathcal{E}$ is precisely the projection of the unstable part of the linearization of (1.1) at u_∞ .

Proof. For fixed $\sigma \in (0, \omega]$ we define the function spaces

$$\begin{aligned}\mathbb{F}_0^l(\sigma) &:= \{f : e^{-\sigma t}f \in L_p(\mathbb{R}_-; X_0^l)\}, \\ \mathbb{F}_1^l(\sigma) &:= \{w : e^{-\sigma t}w \in H_p^1(\mathbb{R}_-; X_0^l) \cap L_p(\mathbb{R}_-; X_1^l)\}, \quad l \in \{c, s, u\},\end{aligned}$$

equipped with their natural norms. We set

$$\mathbb{Y} := \mathbb{Y}(\sigma) := \mathbb{F}_1^c(\sigma) \times \mathbb{F}_1^s(\sigma) \times \mathbb{F}_1^u(\sigma), \quad Z := X_\gamma^u \times X^c,$$

and we define $H_u : B_{\mathbb{Y}}(0, \rho) \times B_Z(0, \rho) \rightarrow \mathbb{Y}$ by

$$H_u((x, y, z), (z_0, \xi))(t) = \begin{bmatrix} x(t) - \int_{-\infty}^t R_c(x(\tau), y(\tau), z(\tau), \xi) d\tau \\ y(t) - \int_{-\infty}^t e^{-A_s(t-\tau)} R_s(x(\tau), y(\tau), z(\tau), \xi) d\tau \\ z(t) - L_u(R_u(x, y, z, \xi), z_0 - \phi_u(\xi)) \end{bmatrix},$$

where $t < 0$ and $w = L_u(f, z_0)$ denotes the unique solution of the backward problem

$$\dot{w}(t) + A_u w(t) = f(t), \quad t \leq 0, \quad w(0) = z_0.$$

Again, we have $H_u(0, 0) = 0$, H_u is of class C^1 w.r.t. (x, y, z, z_0) , but only continuous in ξ , and $D_{(x, y, z)} H_u(0, 0) = I$. As in the previous proof, the implicit function theorem yields a map $\Lambda^u : (z_0, \xi) \mapsto (x, y, z)$ such that $H_u(\Lambda^u(z_0, \xi), (z_0, \xi)) = 0$. Then

$$\lambda^u : (z_0, \xi) \mapsto u(0) = u_* + v(0)$$

yields the foliation of the unstable manifold \mathcal{M}^u near u_* , and

$$\mathcal{M}_\xi^u := \{\lambda^u(z_0, \xi) : z_0 \in B_{X^u}(0, r)\}$$

are the fibers over $B_{X^c}(0, r)$, or equivalently over \mathcal{E} near u_* . Note that the fibers \mathcal{M}_ξ^u are C^1 -manifolds, but they depend only continuously on ξ or equivalently on \mathcal{E} .

The reminder of the proof then follows along similar lines as that of Theorem 3.1 and is therefore left to the interested reader. \square

Note that the fibers can be extended to global fibers following the backward or forward flow as long as it exists. Laterally, i.e. along the direction of \mathcal{E} , the foliation can be extended up to equilibria $u_\# \in \mathcal{E}$ which are no longer normally hyperbolic.

Concerning regularity of the foliation we note that the implicit function theorem yields the following result.

Corollary 4.2. *Under the assumptions of Theorem 3.1 and Theorem 4.1 the following regularity result is valid: if*

$$(A, F) \in C^k(V, \mathcal{B}(X_1, X_0) \times X_0)$$

then the foliations satisfy $\Lambda^l \in C^{k-1}$ and the fibers λ_ξ^l are of class C^k , for all $k \in \mathbb{N} \cup \{\infty, \omega\}$, where $l \in \{s, u\}$ and C^ω means real analytic.

Proof. It follows from the regularity assumptions that $G \in C^k(V, X_0)$. We can then conclude from the second line in (2.13) that $\phi \in C^k(B_{X^c}(0, \rho_0), X_1^s \oplus X_1^u)$. Indeed, for this it suffices to observe that ϕ is implicitly defined by the second line in (2.13), as A_s and A_u are invertible. The assertions follow now from the proof of Theorems 3.1 and 4.1. \square

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